

REGULARITY ESTIMATES FOR CONVEX FUNCTIONS IN CARNOT-CARATHÉODORY SPACES

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ABSTRACT. We prove some regularity estimates for a class of convex functions in Carnot-Carathéodory spaces, generated by Hörmander vector fields. Our approach relies on both the structure of metric balls induced by Hörmander vector fields and upper estimates for subsolutions of sub-Laplacians.

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1. INTRODUCTION

The present paper is devoted to the study of first order regularity properties of convex functions on Carnot-Carathéodory spaces. An important class of these spaces is that of Carnot groups, that can be seen as \mathbb{R}^n equipped with both a group operation and a stratified Lie algebra of left invariant vector fields. By definition, this algebra is spanned by a choice of elements X_1, \dots, X_m , along with their iterated commutators. The latter condition is a special instance of the more general *Hörmander condition* for any given set \mathcal{X} of vector fields. When we only assume that the set \mathcal{X} of vector fields satisfies this condition, we obtain a *Carnot-Carathéodory space*. All linear combinations of elements of \mathcal{X} correspond to the so-called *horizontal vector fields*. These vector fields yield the well known *Carnot-Carathéodory distance*, hence they also generate the metric structure of the space, see Section 2 for precise definitions.

The study of convexity in the sub-Riemannian framework started in Carnot groups, where the seminal works by Danielli, Garofalo and Nhieu [4] and by Lu, Manfredi and Stroffolini [19] have given different approaches. A number of open questions still concern convex functions on Carnot groups and the parallel theory of fully nonlinear subelliptic equations. These facts constitute the main motivations for this study.

Convexity through a second order condition is the approach of [19] and [16]. They introduced the class of *v-convex functions*, as those upper semicontinuous functions $u : \Omega \rightarrow \mathbb{R}$, defined on an open set $\Omega \subset \mathbb{R}^n$, such that

$$(1) \quad \nabla_{\mathcal{X}}^2 u \geq 0 \quad \text{in the viscosity sense,}$$

where the entries of this *horizontal Hessian* in the case u is smooth are exactly the symmetrized second order derivatives $(X_i X_j u + X_j X_i u)/2$ for all $i, j = 1, \dots, m$. For Carnot groups, the vector fields $\mathcal{X} = \{X_1, \dots, X_m\}$ form a basis of the first layer of the stratified Lie algebra.

Among the notions of [4] for convexity in Carnot groups, the *weakly H-convexity* turned out to be the most natural: it requires a convexity-type inequality restricted to some special Lie group combinations of points. This notion can be restated requiring that all restrictions along integral curves of horizontal vector fields are convex. Following this viewpoint one is naturally lead to the local Lipschitz continuity of weakly H-convex functions that are locally bounded from above in Carnot groups, [20]. In fact, the tools needed in the present paper will also require the extension of this theorem to the setting of Hörmander vector fields, according to our Theorem 1.2.

Motivated by the study of comparison principles for Monge-Ampere type equations with respect to Hörmander vector fields, Bardi and Dragoni have introduced a notion of convexity that exactly corresponds to the one dimensional convexity along integral curves of horizontal vector fields, [2]. Precisely, let Ω be an open set of \mathbb{R}^n and let $\mathcal{X} = \{X_1, \dots, X_m\}$ be a set of vector fields defined on \mathbb{R}^n . Then $u : \Omega \rightarrow \mathbb{R}$ is \mathcal{X} -convex, if the composition $u \circ \gamma$ is convex whenever $\gamma : I \rightarrow \Omega$ satisfies $\dot{\gamma} = \sum_{i=1}^m \alpha_i X_i \circ \gamma$ on the open interval I and α_i are arbitrary real numbers.

The v-convexity with respect to a system \mathcal{X} of vector fields simply requires that (1) holds using the vector fields of \mathcal{X} . The main result of [2] is that *in the class of upper semicontinuous functions, v-convexity and \mathcal{X} -convexity do coincide*, where the vector fields of \mathcal{X} are only assumed to be of class C^2 . When these vector fields generate a Carnot group structure, the previous characterization can be found in a number of previous works, [1], [16], [20], [24], [26]. In Theorem 6.1 of [2], it is also proved that \mathcal{X} -convexity, local boundedness and upper semicontinuity imply local Lipschitz continuity with respect to d , where the Carnot-Carathéodory distance d given by \mathcal{X} , is only assumed to yield the Euclidean topology. This result also gives L^∞ -estimates for the horizontal derivatives Xu in terms of the L^∞ -norm of u , where $X \in \mathcal{X}$.

In the case \mathcal{X} generates a Carnot group, these estimates take a quantitative form. Precisely, the central results of [4] and [19, 16] are the following regularity estimates

$$(2) \quad \sup_{w \in B_{x,r}} |u(w)| \leq C_0 \int_{B_{x,2r}} |u(w)| dw$$

$$(3) \quad \text{ess sup}_{w \in B_{x,r}} |\nabla_H u(w)| \leq \frac{C_0}{r} \int_{B_{x,2r}} |u(w)| dw$$

for continuous weakly H-convex functions in [4] and upper semicontinuous v-convex functions in [16], where x varies in \mathbb{G} , $r > 0$ and $C_0 > 0$ is a suitable geometric constant depending on the metric structure of \mathbb{G} . Here $B_{x,r}$ denotes the metric ball with respect to the homogeneous distance fixed on the group and $\nabla_H u$ is the *horizontal gradient* ($X_1 u, \dots, X_m u$). We wish to emphasize the important role of estimates (2) and (3) in the study of fine properties of convex functions in Carnot groups, especially in relation with second order differentiability and distributional characterizations.

Our main result establishes that the previous estimates can be suitably extended to Carnot-Carathéodory spaces generated by a set \mathcal{X} of Hörmander vector fields. Precisely, we have the following theorem.

Theorem 1.1. *Let $\mathcal{X} = \{X_1, \dots, X_m\}$ be a set of Hörmander vector fields, let $\Omega \subset \mathbb{R}^n$ be open and let $K \subset \Omega$ be compact. Then there exist $C > 0$ and $R > 0$, depending on K , such that each \mathcal{X} -convex function $u : \Omega \rightarrow \mathbb{R}$, that is locally bounded from above, for every $x \in K$ satisfies the following estimates*

$$(4) \quad \sup_{B_{x,r}} |u| \leq C \int_{B_{x,2r}} |u(w)| dw$$

$$(5) \quad |u(y) - u(z)| \leq C \frac{d(y,z)}{r} \int_{B_{x,2r}} |u(w)| dw,$$

for every $0 < r < R$ and every $y, z \in B_{x,r}$.

Clearly, the constant $C > 0$ cannot be chosen independent of K as in the case of Carnot groups, since general Carnot-Carathéodory spaces need not have either a group operation or dilations and the *doubling dimension* may change from point to point. This should suggest that the estimates of Theorem 1.1 are somehow sharp.

Our approach to prove (4) and (5) differs from both the geometric approach of [4] and the PDEs approach of [19, 16]. In fact, we need both these aspects, according to the following scheme. We start from a \mathcal{X} -convex function $u : \Omega \rightarrow \mathbb{R}$ that is locally bounded from above. By a result of D. Morbidelli, [21], the Carnot-Carathéodory ball can be covered by suitable compositions of flows of horizontal vector fields in a quantitative way, depending on the radius of the ball. Up to some more technical facts, this essentially allows us to apply the approach of [20] that relies on the one dimensional convexity of u along these flows, hence obtaining explicit Lipschitz estimates. Precisely, we have the following result.

Theorem 1.2. *Let $\mathcal{X} = \{X_1, \dots, X_m\}$ be a set of Hörmander vector fields, let $\Omega \subset \mathbb{R}^n$ be open and let $u : \Omega \rightarrow \mathbb{R}$ be a \mathcal{X} -convex function that is locally bounded from above. It follows that u is locally Lipschitz continuous. More precisely, if $K \subset \Omega$ is compact and $0 < r < \text{dist}_d(K, \Omega^c)$, then for every $x, y \in K$ we have*

$$(6) \quad |u(x) - u(y)| \leq \frac{C}{r} d(x, y) \sup_{K_r} |u|,$$

where $K_r = \{z \in \mathbb{R}^n : \text{dist}_d(K, z) \leq r\} \subset \Omega$ and $C > 0$ only depends on K and \mathcal{X} .

It follows that u belongs to the anisotropic Sobolev space $W_{\mathcal{X}, \text{loc}}^{1,2}(\Omega)$, see Section 2 for more information. The crucial step is to show that for every $x \in \Omega$ the \mathcal{X} -convex function u is a *weak subsolutions* of a suitable “pointed sub-Laplacian”

$$\mathcal{L}_x = \sum_{j=1}^m Y_j^2,$$

that is constructed around x . This step is precisely stated in the following theorem.

Theorem 1.3. *Let $\mathcal{X} = \{X_1, \dots, X_m\}$ be a set of Hörmander vector fields, let $\Omega \subset \mathbb{R}^n$ be open, let $x_0 \in \Omega$ and let $u : \Omega \rightarrow \mathbb{R}$ be a \mathcal{X} -convex function that is locally bounded from above. There exist $\delta_0 > 0$ and a family of vector fields $\mathcal{X}_1 = \{Y_1, \dots, Y_m\}$, with $Y_i = \sum_{j=1}^m a_{ij} X_j$, and $a_{ij} \in \{0, 1\}$, both depending on x_0 , such that $B_{x_0, \delta_0} \subset \Omega$ and u is a weak subsolution of the equation*

$$(7) \quad \sum_{i=1}^m Y_i^2 v = 0 \quad \text{on} \quad B_{x_0, \delta_0}.$$

Since the Lebesgue measure is locally doubling with respect to metric balls and the Poincaré inequality holds, the classical Moser iteration technique holds for weak subsolutions to the sub-Laplacian equation, hence getting the classical inequality

$$(8) \quad \sup_{B_{y, \frac{r}{2}}} u \leq \kappa_x \int_{B_{y, r}} |u(z)| dz$$

for $0 < r < \sigma_x$ and $y \in B_{x, \delta_x}$, where the positive constants κ_x , σ_x and $\delta_x > 0$ depend on x , see Section 5 for more information and in particular Corollary 5.4. The lower

estimate of u is reached using again the approximate exponential, hence obtaining the following pointwise estimate

$$(9) \quad 2^{N_x} u(x) - (2^{N_x} - 1) \sup_{B_{x, \bar{N}\delta}} u \leq \inf_{B_{x, b\delta}} u,$$

where N_x depends on x and it satisfies the uniform inequality $1 \leq N_x \leq \bar{N}$ on some compact set, see Lemma 4.1. This eventually leads us to the proof of (4). The estimate (5) is a straightforward consequence of Theorem 1.2 joined with Theorem 6.2. In sum, the geometric part of our method arises from a quantitative representation of the Carnot-Caratéodory ball by approximates exponentials and it leads us to the lower estimates. The PDEs part of our approach provides the upper estimates.

At this point, it is worth to mention that another approach to the regularity of convex functions with respect to general Hörmander vector fields has been also considered by Trudinger in [25]. Recall that a smooth k -convex function has the property that all j th elementary symmetric functions of the horizontal Hessian $\nabla_{\mathcal{X}}^2 u$ are nonnegative for all $j = 1, \dots, k$ and $k \leq m$, where $\mathcal{X} = \{X_1, \dots, X_m\}$. In [25], nonsmooth k -convex functions are defined by L^1_{loc} -limits of smooth k -convex functions and in the works by Gutierrez and Montanari, [12, 13], these functions are introduced as locally uniform limits of smooth k -convex functions.

On the other hand, these two different topological closures yield the same class of functions. This is a consequence of the Hölder continuity estimates established by Trudinger for all nonsmooth k -convex with $1 \leq k \leq m$, [25]. We finally notice that Theorem 1.1 provides the Lipschitz regularity of all nonsmooth m -convex functions.

The present paper is structured as follows. Section 2 recalls the main notions and the basic facts. Section 3 presents some basic properties of the “approximate exponential”, restating some results of [21] in view of our applications. Section 4 contains the proof of Theorem 1.2, according to which \mathcal{X} -convex functions that are locally bounded from above are also locally Lipschitz. In Section 5, we state the local integral upper bounds for weak subsolutions of sub-Laplacians and prove Theorem 1.3. Section 6 collects the preceding results in order to establish (4) of Theorem 1.1.

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2. SOME BASIC NOTIONS AND FACTS

Throughout the paper, we consider a family \mathcal{X} of smooth vector fields X_1, \dots, X_m on \mathbb{R}^n , which satisfy the *Hörmander condition*: for every $x \in \mathbb{R}^n$ there exists a positive integer r' such that

$$(10) \quad \text{span}\{X_{[S]}(x) : |S| \leq r'\} = \mathbb{R}^n.$$

For every multi-index $S = (s_1, \dots, s_p) \in \{1, 2, \dots, m\}^p$, we have set $|S| = p$ and

$$(11) \quad X_{[S]} = [X_{s_1}, [\dots, [X_{s_{p-1}}, X_{s_p}] \dots]].$$

Remark 2.1. As a consequence of the Hörmander condition, for every bounded set $A \subset \mathbb{R}^n$ we have a positive integer r such that (10) is satisfied for $r' = r$ and all $x \in A$.

Definition 2.2 (Flow of a vector field). Let X be a smooth vector field of \mathbb{R}^n and let $x \in \mathbb{R}^n$. We consider the Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = x \end{cases}$$

and denote its solution by $t \rightarrow \Phi^X(x, t)$. The mapping Φ^X defined on an open neighbourhood of $\mathbb{R}^n \times \{0\}$ in \mathbb{R}^{n+1} is the flow associated to X . The flow Φ^X will also define the local diffeomorphism $\Phi_t^X(\cdot) = \Phi^X(\cdot, t)$ on bounded open sets for t sufficiently small.

Definition 2.3 (CC-distances and metric balls). For every $x, y \in \mathbb{R}^n$ we define the following distance

$$(12) \quad d(x, y) = \inf\{t > 0 : \text{there exists } \gamma \in \Gamma_{x,y}(t)\},$$

where $\Gamma_{x,y}(t)$ denotes the family of all absolutely continuous curves $\gamma : [0, t] \rightarrow \mathbb{R}^n$ with $\gamma(0) = x$, $\gamma(t) = y$ and such that for a.e. $s \in [0, t]$ we have

$$\dot{\gamma}(s) = \sum_{j=1}^m a_j(s) X_j(\gamma(s)) \quad \text{and} \quad \max_{1 \leq j \leq m} |a_j(s)| \leq 1.$$

This distance along with its properties can be found in [23]. If the previous condition is modified replacing $\max_{1 \leq j \leq m} |a_j(s)|$ with $(\sum_{1 \leq j \leq m} a_j(s)^2)^{1/2}$, then in the context of PDEs this distance first appeared in a work by Fefferman and Phong, [5]. Metric balls are defined using the following notation

$$B_{x,r} = \{z \in \mathbb{R}^n : d(z, x) < r\}, \quad D_{x,r} = \{z \in \mathbb{R}^n : d(z, x) \leq r\}$$

for any $r > 0$ and $x \in \mathbb{R}^n$. We say that d is the Carnot-Carathéodory distance, in short *CC-distance*, with respect to \mathcal{X} . Another analogous distance that will be important for the sequel is the following one. Let $\Gamma_{x,y}^c(t)$ be the family of all absolutely continuous curves $\gamma : [0, t] \rightarrow \mathbb{R}^n$ with $\gamma(0) = x$, $\gamma(t) = y$, such that for a.e. $s \in [0, t]$ we have

$$\dot{\gamma}(s) = \sum_{j=1}^m a_j(s) X_j(\gamma(s)) \quad \text{and} \quad (a_1, \dots, a_m) \in \{\pm e_1, \dots, \pm e_m\},$$

where the curve (a_1, \dots, a_m) is piecewise constant on $[0, t]$ and (e_1, \dots, e_m) is the canonical basis of \mathbb{R}^m . Thus, we define the distance

$$(13) \quad \rho(x, y) = \inf\{t > 0 : \text{there exists } \gamma \in \Gamma_{x,y}^c(t)\}.$$

This distance in the framework of PDEs has been first introduced by Franchi and Lanconelli, [6], [17], [7].

Remark 2.4. Let us consider $X \in \mathcal{X}$ and $t, \tau \in \mathbb{R}$, by definition of d and ρ , we have

$$\max\{d(\Phi_t^X(x), \Phi_\tau^X(x)), \rho(\Phi_t^X(x), \Phi_\tau^X(x))\} \leq |t - \tau|$$

for any $x \in \mathbb{R}^n$, whenever the flows are defined for times t and τ .

Remark 2.5. Let \mathcal{X} be the family of smooth Hörmander vector fields X_1, \dots, X_m introduced in Section 2. Then by a rescaling argument, one can easily check that there holds

$$(14) \quad d(x, y) = \inf \left\{ \delta > 0 : \text{there exists } \gamma \in \Gamma_{x,y}^\delta \right\},$$

where $\Gamma_{x,y}^\delta(\mathcal{X})$ is the family of absolutely continuous curves $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(1) = y$ and for a.e. $t \in [0, 1]$ we have

$$\dot{\gamma}(t) = \sum_{j=1}^m a_j(t) X_j(\gamma(t)) \quad \text{and} \quad \max_{1 \leq j \leq m} |a_j(t)| < \delta,$$

where d is introduced in Definition 2.3.

Lemma 2.6. Let d and d_1 two CC-distances associated to the families of smooth Hörmander vector fields $\mathcal{X} = \{X_1, \dots, X_m\}$ and $\mathcal{X}_1 = \{Y_1, \dots, Y_m\}$, respectively. Let $\{i_1, j_1, \dots, j_{m-1}\} = \{1, 2, \dots, m\}$ and assume that $Y_j = X_j$ for all $j \neq i_1$ and $Y_{i_1} = X_{i_1} + X_{j_1}$. Then we have $4^{-1}d \leq d_1 \leq 4d$.

Proof. We can use for d and d_1 the equivalent definition stated in Remark 2.5. Taking this into account, we fix a compact set $K \subset \mathbb{R}^n$ and choose any $x_1, x_2 \in K$, setting $d(x_1, x_2) = \delta/2$, for some $\delta > 0$. Then there exists an absolutely continuous curve $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ belonging to $\Gamma_{x,y}^\delta(\mathcal{X})$. Clearly, we observe that

$$\dot{\gamma} = a_{i_1} Y_{i_1}(\gamma) + (a_{j_1} - a_{i_1}) Y_{j_1}(\gamma) + \sum_{s=2}^{m-1} a_{j_s} Y_{j_s}(\gamma),$$

hence $\gamma \in \Gamma_{x,y}^{2\delta}(\mathcal{X}_1)$, then $d_1(x, y) \leq 2\delta = 4d(x, y)$. In analogous way we get $d(x_1, x_2) \leq 4d_1(x_1, x_2)$, concluding the proof. \square

Next, we introduce the anisotropic Sobolev space $W_{\mathcal{X}}^{1,p}$ with respect to the family \mathcal{X} . Throughout, for every open set $\Omega \subset \mathbb{R}^n$ we denote by $C_c^\infty(\Omega)$, the class of smooth functions with compact support.

Definition 2.7. Given an open set $\Omega \subset \mathbb{R}^n$, we define the \mathcal{X} -Sobolev space $W_{\mathcal{X}}^{1,p}(\Omega)$, with $1 \leq p \leq \infty$, as follows

$$W_{\mathcal{X}}^{1,p}(\Omega) = \{f \in L^p(\Omega), X_j f \in L^p(\Omega), j = 1, \dots, m\},$$

where $X_j u$ is the *distributional derivative* of $u \in L_{loc}^1(\Omega)$, namely

$$\langle X_i u, \phi \rangle = \int_{\Omega} u X_i^* \phi \, dx, \quad \phi \in C_0^\infty(\Omega),$$

and X_i^* is the formal adjoint of X_i , namely, $X_i^* = -X_i - \operatorname{div} X_i$.

The linear space $W_{\mathcal{X}}^{1,p}(\Omega)$ is turned into a Banach space by the norm

$$\|f\|_{W_{\mathcal{X}}^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \sum_{j=1}^m \|X_j f\|_{L^p(\Omega)}.$$

A function $u \in W_{\mathcal{X}}^{1,2}(\Omega)$ is an \mathcal{L} -weak subsolution of

$$(15) \quad \mathcal{L}u = \sum_{i=1}^m X_i^2 u = 0,$$

if for every nonnegative $\eta \in W_{\mathcal{X},0}^{1,2}(\Omega)$, we have $\sum_{i=1}^m \int_{\Omega} X_i u X_i^* \eta dx \geq 0$.

Lemma 2.8. *Let Ω' be an open set compactly contained in Ω and let $X \in \mathcal{X}$. There exists $T > 0$ such that the map Φ^X is well defined on $\Omega' \times (-2T, 2T)$ and for every $t \in (-2T, 2T)$, the mapping $\Phi^X(\cdot, t) : \Omega' \rightarrow \mathbb{R}^n$ is bi-Lipschitz onto its image with inverse $\Phi^X(\cdot, -t)$. The Jacobian J_X of Φ^X satisfies*

$$J_X(x, t) = 1 + \tilde{J}_X(x, t) \quad \text{and} \quad |\tilde{J}_X(x, t)| \leq C|t|$$

for all $x \in \Omega'$ and $|t| < 2T$, where $C > 0$ is independent of x and t .

The proof of this lemma can be achieved by standard ODEs methods, see also [9] for the general case of a Lipschitz vector field.

Theorem 2.9. *Every Lipschitz function on an open set $\Omega \subset \mathbb{R}^n$ belongs to $W_{\mathcal{X}}^{1,\infty}(\Omega)$.*

The proof of Theorem 2.9 can be found from either Proposition 2.9 of [9] or Theorem 1.3 of [11]. From either these papers or the arguments of Theorem 11.7 of [14], it is also not difficult to deduce the following proposition.

Proposition 2.10. *Let $u : \Omega \rightarrow \mathbb{R}$ be a Lipschitz function. Let X be a vector field of \mathcal{X} and fix $x \in \Omega$. Let $\Phi_t^X(x)$ be the flow of X starting at x . Then the directional derivative $\frac{d}{dt}u(\Phi_t^X(x))|_{t=0}$ exists almost everywhere and it coincides with the distributional derivative Xu .*

3. APPROXIMATE EXPONENTIALS AND CC-DISTANCES

In this section, we introduce a class of exponential mappings for vector fields and recall their properties, following notations and results by D. Morbidelli [21]. We define

$$\begin{aligned} \mathcal{X}^{(1)} &= \{X_1, \dots, X_m\}, \\ \mathcal{X}^{(2)} &= \{X_{[i_1, i_2]}, \ 1 \leq i_1 < i_2 \leq m\} \end{aligned}$$

and so on, in such a manner that elements of $\mathcal{X}^{(k)}$ are the commutators of length k . We denote by Y_1, \dots, Y_q an enumeration of all the elements of $\mathcal{X}^{(1)}, \dots, \mathcal{X}^{(r)}$, where r is an integer large enough to ensure that Y_1, \dots, Y_q span \mathbb{R}^n at each point of a fixed bounded open set $\Omega \subset \mathbb{R}^n$, see Remark 2.1. We call r the *local spanning step* and q

the *local spanning number* of \mathcal{X} , to underly that they depend on Ω . It may be worth to stress that the Lie algebra spanned by \mathcal{X} at some point need not be nilpotent, although the local spanning step is finite.

If Y_i is an element of $\mathcal{X}^{(j)}$, we say Y_i has formal degree $d_i := d(Y_i) = j$. Let $I = (i_1, \dots, i_n) \in \{1, 2, \dots, q\}^n$ be a multi-index and define from [23] the functions

$$\lambda_I(x) = \det[Y_{i_1}(x), \dots, Y_{i_n}(x)] \quad \text{and} \quad \|h\|_I = \max_{j=1, \dots, n} |h_j|^{\frac{1}{d(Y_{i_j})}}.$$

As a consequence of the choice of (Y_1, \dots, Y_q) , we have that for every $x \in \Omega$ there exists $I \in \{1, 2, \dots, q\}^n$ with $\lambda_I(x) \neq 0$. We denote by $d(I)$ the integer $d_{i_1} + \dots + d_{i_n}$, where $d_{i_k} = d(Y_{i_k})$.

Definition 3.1. Let $X, S \in \mathcal{X}$ and consider the mappings Φ_t^X and Φ_t^S , that coincide with Φ_1^{tX} and Φ_1^{tS} , respectively. Thus, for t sufficiently small, we can define the *local exponentials* $\exp(tX) := \Phi_1^{tX}$ and $\exp(tS) := \Phi_1^{tS}$, along with the *local product*

$$\exp(tX) \exp(tS) = \Phi_1^{tX} \circ \Phi_1^{tS}.$$

Let S_1, \dots, S_l be vector fields belonging to the family \mathcal{X} . Therefore, for every $a \in \mathbb{R}$ sufficiently small, we can define

$$\begin{aligned} C_1(a, S_1) &= \exp(aS_1), \\ C_2(a, S_1, S_2) &= \exp(-aS_2) \exp(-aS_1) \exp(aS_2) \exp(aS_1), \\ C_l(a, S_1, \dots, S_l) &= C_{l-1}(a; S_2, \dots, S_l)^{-1} \exp(-aS_1) C_{l-1}(a; S_2, \dots, S_l) \exp(aS_1). \end{aligned}$$

According to (14) of [21], for $\sigma \in \mathbb{R}$ sufficiently small we define

$$(16) \quad e_{\text{ap}}^{\sigma S_{[1, \dots, l]}} = \begin{cases} C_l(\sigma^{\frac{1}{l}}, S_1, \dots, S_l), & \sigma > 0, \\ C_l(|\sigma|^{\frac{1}{l}}, S_1, \dots, S_l)^{-1}, & \sigma < 0. \end{cases}$$

Following (16) of [21], given a multi-index $I = (i_1, \dots, i_n)$, $1 \leq i_j \leq q$ and $h \in \mathbb{R}^n$ small enough, we also set

$$(17) \quad E_I(x, h) = e_{\text{ap}}^{h_1 Y_{i_1}} \dots e_{\text{ap}}^{h_n Y_{i_n}}(x).$$

The next theorem, that is contained in Theorem 3.1 of [21], shows that the approximate exponential maps give a good representation of the Carnot-Carathéodory balls.

Theorem 3.2. *If $\Omega \subset \mathbb{R}^n$ is an open bounded set with local spanning number q and $K \subset \Omega$ is a compact set, then there exist $\delta_0 > 0$ and positive numbers a and b , $b < a < 1$, so that, given any $I \in \{1, \dots, q\}^n$ such that*

$$(18) \quad |\lambda_I(x)| \delta^{d(I)} \geq \frac{1}{2} \max_{J \in \{1, \dots, q\}^n} |\lambda_J(x)| \delta^{d(J)},$$

for $x \in K$ and $0 < \delta < \delta_0$, it follows that $B_{x, b\delta} \subset E_{I, x}(\{h \in \mathbb{R}^n : \|h\|_I < a\delta\}) \subset B_{x, \delta}$.

Following the terminology of [23], we introduce the following definition.

Definition 3.3. We say that two distances ρ_1 and ρ_2 in \mathbb{R}^n are *equivalent*, if for every compact set $K \subset \mathbb{R}^n$, there exist $c_K \geq 1$, depending on K , such that

$$c_K^{-1} \rho_1(x, y) \leq \rho_2(x, y) \leq c_K \rho_1(x, y) \quad \text{for all } x, y \in K.$$

Remark 3.4. We have stated Theorem 3.2 using only metric balls with respect to the distance d . In fact, in [21] the same symbol denotes the same distance, with a different definition, see Remark 2.5. Up to a change of the constant $b > 0$ in Theorem 3.1 of [21], we can replace the distance denoted by " ρ " in [21] with d . In fact, these two distances are equivalent, due to Theorem 4 of [23], joined with our Remark 2.5.

The following proposition has been pointed out to us by D. Morbidelli. It is a consequence of the seminal paper by A. Nagel, E. M. Stein and S. Wainger [23], and it can be also found as a consequence of Theorem 3.1 of [21].

Proposition 3.5. *The distances d and ρ introduced in Definition 2.3 are equivalent.*

Remark 3.6. Notice that the inequality $d \leq \rho$ is trivial. As a consequence of the previous proposition, \mathcal{X} -convex functions that are locally bounded are also locally Lipschitz continuous with respect to d and any other equivalent distance, according to the notion of equivalence given in Definition 3.3

We fix a multi-index $I = (i_1, \dots, i_n)$ and for each Y_{i_k} we have a multi index

$$J_{i_k} = (j_1^{i_k}, j_2^{i_k}, \dots, j_{d_{i_k}}^{i_k}) \quad \text{such that} \quad Y_{i_k} = X_{[J_{i_k}]},$$

where d_{i_k} is the formal degree of Y_{i_k} . We notice that $1 \leq j_s^{i_k} \leq m$ for all $1 \leq s \leq d_{i_k}$ and $d_{i_k} \leq r$, where r is the local spanning step of \mathcal{X} . By definition of e_{ap} we get

$$(19) \quad e_{\text{ap}}^{hY_{i_k}} = \begin{cases} \prod_{s=1}^{N_{i_k}} \exp(\sigma_s h^{\frac{1}{d_{i_k}}} X_s^{i_k}) & h \geq 0, \\ \prod_{s=1}^{N_{i_k}} \exp(-\sigma_{N_{i_k}+1-s} |h|^{\frac{1}{d_{i_k}}} X_{N_{i_k}+1-s}^{i_k}) & h < 0. \end{cases}$$

where $\sigma_s \in \{-1, 1\}$, N_{i_k} is the length of $e_{\text{ap}}^{hY_{i_k}}$ and $X_1^{i_k}, X_2^{i_k}, \dots, X_{N_{i_k}}^{i_k}$ is a suitable possibly iterated choice among the vectors $X_{j_1^{i_k}}^{i_k}, X_{j_2^{i_k}}^{i_k}, \dots, X_{j_{d_{i_k}}^{i_k}}^{i_k}$. A simple calculation gives $N_{i_k} = 2^{d_{i_k}} - 2 + 2^{d_{i_k}-1}$. We define $N(I) = \sum_{k=1}^n 2N_{i_k}$ along with the mapping $G_{I,x} : \mathbb{R}^N \rightarrow \mathbb{R}^n$, that is

$$(20) \quad G_{I,x}(w) = \prod_{k=1}^n \left\{ \prod_{s=1}^{N_{i_k}} \exp(w_{k,s,2} X_{N_{i_k}+1-s}^{i_k}) \prod_{s=1}^{N_{i_k}} \exp(w_{k,s,1} X_s^{i_k}) \right\} (x).$$

In the definition of $G_{I,x}$, we use the product to indicate the composition of flows according to the order that starts from the right. The variable w denotes the vector

$$(w_{1,1,1}, w_{1,2,1}, \dots, w_{1,N_{i_1},1}, w_{1,1,2}, w_{1,2,2}, \dots, w_{1,N_{i_1},2}, \dots, w_{n,1,2}, \dots, w_{n,N_{i_n},2})$$

belonging to $\mathbb{R}^{N(I)}$. The integer $N(I)$ is locally uniformly bounded from above, since every multi-index $I = (i_1, \dots, i_n)$ of Theorem 3.2 depends on x and satisfies $N_{i_k} \leq$

$2^r - 2 + 2^{r-1}$, where r is the local spanning step of \mathcal{X} , depending on the fixed bounded open set Ω . Therefore we have a local upper bound \bar{N} defined as follows

$$(21) \quad \bar{N} = 2n(2^{r+1} - 2 + 2^{r-1})$$

and clearly $N(I) \leq \bar{N}$, where \bar{N} is independent of I .

Definition 3.7. For every $N \in \mathbb{N} \setminus \{0\}$, we set $\|w\|_N = \max_{k=1, \dots, N} |w_k|$, for every $w \in \mathbb{R}^N$.

The corresponding open ball is defined as follows

$$S_{N,\delta} = \{w \in \mathbb{R}^N : \|w\|_N < \delta\}.$$

From standard theorems on ODEs, one can establish the following fact.

Proposition 3.8. *If $K \subset \Omega$ is a compact set and $N \in \mathbb{N}$ is positive, then there exists $\delta_1 > 0$ only depending on K , Ω and \mathcal{X} such that for every $0 < \delta \leq \delta_1$ and every $x \in K$ we have $B_{x,N\delta_1} \subset \Omega$ and for every integers $1 \leq j_1, \dots, j_N \leq m$, the composition*

$$\left(\exp(w_N X_{j_N}) \cdots \exp(w_2 X_{j_2}) \exp(w_1 X_{j_1}) \right)(x)$$

is well defined and contained in $B_{x,N\delta}$ for all $w \in S_{N,\delta}$.

The previous proposition immediately leads us to the following consequence.

Corollary 3.9. *Let Ω be an open bounded set with local spanning number q and local spanning step r . If $K \subset \Omega$ is a compact set, then there exist $\delta_1 > 0$ such that for every $x \in K$, every $0 < \delta \leq \delta_1$ and every multi-index $I \in \{1, 2, \dots, q\}^n$, the mapping $G_{I,x}$ introduced in (20) is well defined on $S_{N(I),\delta}$ and*

$$G_{I,x}(S_{N(I),\delta}) \subset B_{x,\bar{N}\delta} \subset D_{x,\bar{N}\delta_1} \subset \Omega,$$

where \bar{N} is defined in (21).

For any of the above multi-indexes $I = (i_1, \dots, i_n)$, we introduce the function $F_{I,x} : \mathbb{R}^n \rightarrow \mathbb{R}^{N(I)}$ as follows

$$\begin{aligned} F_{I,x}(h_1, \dots, h_n) = & (\sigma_{1,1} \delta_1(h_1) h_1^{\frac{1}{d_{i_1}}}, \dots, \sigma_{1,N_{i_1}} \delta_1(h_1) h_1^{\frac{1}{d_{i_1}}}, -\sigma_{1,N_{i_1}} \delta_2(h_1) |h_1|^{\frac{1}{d_{i_1}}}, \\ & \dots, -\sigma_{1,1} \delta_2(h_1) |h_1|^{\frac{1}{d_{i_1}}}, \dots, \sigma_{n,1} \delta_1(h_n) h_n^{\frac{1}{d_{i_n}}}, \dots, \sigma_{n,N_{i_n}} \delta_1(h_n) h_n^{\frac{1}{d_{i_n}}}, \dots) \end{aligned}$$

where $\sigma_{k,j} \in \{-1, 1\}$, $k = 1, \dots, n$ and $j = 1, \dots, N_{i_k}$. More precisely, we have

$$(22) \quad F_{I,x}(h) = \sum_{k=1}^n \left\{ \sum_{s=1}^{N_{i_k}} \sigma_{k,s} \delta_1(h_k) h_k^{1/d_{i_k}} e_{k,s,1} - \sum_{s=1}^{N_{i_k}} \sigma_{k,N_{i_k}+1-s} \delta_2(h_k) |h_k|^{1/d_{i_k}} e_{k,s,2} \right\},$$

where we have introduced the canonical basis

$$\{e_{k,s,i} : 1 \leq k \leq n, 1 \leq s \leq N_{i_k}, i = 1, 2\}$$

of $\mathbb{R}^{N(I)}$ and the functions

$$\delta_1(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad \text{and} \quad \delta_2(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases}.$$

Remark 3.10. From the definitions of $G_{I,x}$ and $F_{I,x}$, it is straightforward to observe that $E_{I,x} = G_{I,x} \circ F_{I,x}$ on a sufficiently small neighbourhood of the origin in \mathbb{R}^n .

Theorem 3.11. *If $\Omega \subset \mathbb{R}^n$ is an open bounded set with local spanning number q and $K \subset \Omega$ is compact, then there exist $\delta_0 > 0$ and positive numbers a and b , $b < a < 1$, so that for any $x \in K$ and $0 < \delta < \delta_0$ and any $I \in \{1, \dots, q\}^n$ with*

$$(23) \quad |\lambda_I(x)|\delta^{d(I)} \geq \frac{1}{2} \max_{J \in \{1, \dots, q\}^n} |\lambda_J(x)|\delta^{d(J)},$$

we have $B_{x,b\delta} \subset G_{I,x}(\{w \in \mathbb{R}^{N(I)} : \|w\|_{N(I)} < a\delta\}) \subset B_{x,\bar{N}\delta} \subset D_{x,\bar{N}\delta_0} \subset \Omega$.

Proof. From Theorem 3.2, we get the existence of $\delta_0, a, b > 0$, with $b < a < 1$ such that for every $x \in K$, $0 < \delta < \delta_0$ and $I \in \{1, \dots, q\}^n$ satisfying (23), we have the inclusion

$$B_{x,b\delta} \subset E_{I,x}(\{h \in \mathbb{R}^n : \|h\|_I < a\delta\}).$$

This proves the validity of this inclusion, since for every $x \in K$ and $0 < \delta < \delta_0$ the existence of I satisfying (23) is trivial. From formula (22), we have

$$(24) \quad \|F_{I,x}(h)\|_{N(I)} = \|h\|_I \quad \text{for all } h \in \mathbb{R}^n.$$

Remark 3.10 implies that $E_{I,x}(h) = G_{I,x} \circ F_{I,x}(h)$ for all $h \in \mathbb{R}^n$, possibly small, such that $G_{I,x}$, introduced in (20), is well defined on $F_{I,x}(h)$. In view of Corollary 3.9, it is not restrictive to choose $\delta_0 > 0$ possibly smaller, such that $G_{I,x}$ is well defined on

$$(25) \quad S_{N(I),\delta_0} \quad \text{and} \quad G_{I,x}(S_{N(I),\delta}) \subset B_{x,\bar{N}\delta} \subset D_{x,\bar{N}\delta_0} \subset \Omega.$$

Taking into account (24), we have $F_{I,x}(\{h \in \mathbb{R}^n : \|h\|_I < a\delta\}) \subset S_{N(I),\delta}$, that leads us to the following inclusions

$$(26) \quad B_{x,b\delta} \subset E_{I,x}(\{h \in \mathbb{R}^n : \|h\|_I < a\delta\}) \subset G_{I,x}(S_{N(I),\delta}) \subset B_{x,\bar{N}\delta}$$

concluding the proof. \square

According to [23], for $x \in \mathbb{R}^n$, we set

$$\Lambda(x, \delta) = \sum_{I \in \{1, 2, \dots, q\}^n} |\lambda_I(x)| \delta^{d(I)}.$$

From Theorem 1 of [23], we get the following important fact.

Theorem 3.12. *For every $K \subset \mathbb{R}^n$ compact, there exist $\delta_0 > 0$ and positive constants C_1 and C_2 , depending on K , so that for all $x \in K$ and every $0 < \delta < \delta_0$ we have*

$$C_1 \leq \frac{|B_{x,\delta}|}{\Lambda(x, \delta)} \leq C_2.$$

The point of this theorem is that it gives the doubling property of metric balls, as pointed out in [23]. In fact, Λ is a polynomial with respect to δ , that only depends on the enumeration of vector fields Y_1, \dots, Y_q on some fixed open bounded set Ω . Thus, we have the following corollary.

Corollary 3.13. *For every compact set $K \subset \mathbb{R}^n$ there exist positive constants C and r_0 , depending on K , such that for every $x \in K$ and every $0 < r < r_0$, we have*

$$|B_{x,2r}| \leq C |B_{x,r}|.$$

4. BOUNDEDNESS FROM ABOVE IMPLIES LIPSCHITZ CONTINUITY

This section is devoted to the proof of the local Lipschitz continuity of \mathcal{X} -convex functions that are locally bounded from above. Precisely, we will prove Theorem 1.2.

Lemma 4.1. *Let $u : \Omega \rightarrow \mathbb{R}$ be a \mathcal{X} -convex function on an open set $\Omega \subset \mathbb{R}^n$ and let K be a compact set. Then there exist $\delta_0 > 0$, $0 < b < 1$ and an integer \bar{N} only depending on K and \mathcal{X} such that for every $x \in K$, there exists an integer $1 \leq N_x \leq \bar{N}$ such that for every $0 < \delta < \delta_0$ we have $D_{x,\bar{N}\delta_0} \subset \Omega$ and*

$$(27) \quad 2^{N_x} u(x) - (2^{N_x} - 1) \sup_{B_{x,\bar{N}\delta}} u \leq \inf_{B_{x,b\delta}} u.$$

Proof. Let Ω' be an open bounded set containing K such that $\overline{\Omega'} \subset \Omega$, let r be the local spanning step and q be the local spanning number with local spanning frame Y_1, \dots, Y_q on Ω' . We apply Theorem 3.11 to both K and Ω' , getting an integer \bar{N} and positive number $\delta_0 > 0$, $0 < b < a < 1$, depending on K , Ω' and \mathcal{X} , having the properties stated in this theorem. Thus, we choose any $x \in K$ and $0 < \delta < \delta_0$, so that we can find a multi-index $I \in \{1, \dots, q\}^n$ such that (23) holds. Theorem 3.11 implies that

$$B_{x,b\delta} \subset G_{I,x}(S_{N(I),a\delta}) \subset B_{x,\bar{N}\delta} \subset D_{x,\bar{N}\delta_0} \subset \Omega'$$

where $G_{I,x}$ is defined in (20). In particular, the closure $\overline{S_{N(I),x}}$ satisfies

$$G_{I,x}(\overline{S_{N(I),a\delta}}) \subset \Omega'.$$

Let us consider the scalar function $\varphi(w) = u \circ G_{I,x}(w)$, that is well defined for all $w \in \overline{S_{N(I),a\delta}}$. By definition of \mathcal{X} -convexity, we have

$$\mu_1 = 2\varphi(0) - \sup_{B_{x,\bar{N}\delta}} u \leq 2\varphi(0) - \varphi(-w_1, 0, \dots, 0) \leq \varphi(w_1, 0, \dots, 0),$$

whenever $|w_1| \leq a\delta$. Notice that $\mu_1 = 2u(x) - \sup_{B_{x,\bar{N}\delta}} u$. Of course, in the case $\sup_{B_{x,\bar{N}\delta}} u = +\infty$, then the inequalities (29) become trivial. For each $w_1 \in [-a\delta, a\delta]$, the function

$$[-a\delta, a\delta] \ni s \mapsto \varphi(w_1, s, 0, \dots, 0),$$

is convex with respect to s , hence arguing as before we get

$$\mu_2 = 2\mu_1 - \sup_{B_{x,\bar{N}\delta}} u \leq \varphi(w_1, s, 0, \dots, 0).$$

whenever $|s| \leq a\delta$. We can repeat this argument up to $N(I)$ times, achieving

$$(28) \quad \mu_{N(I)} \leq u \circ G_{I,x_0}(w) \quad \text{for every } w \in \bar{S}_{N(I),a\delta},$$

where $\mu_j = 2\mu_{j-1} - \sup_{B_{x,\bar{N}\delta}} u$ for $j = 1, \dots, N(I)$. In particular, we have

$$\mu_{N(I)} = 2^{N(I)}u(x) - \left(\sum_{j=0}^{N(I)-1} 2^j \right) M = 2^{N(I)}u(x) - 2^{N(I)}M + M$$

with $M = \sup_{B_{x,\bar{N}\delta}} u$. In sum, we have proved that there exist $\delta_0 > 0$, $0 < b < 1$ and an integer \bar{N} only depending on K and \mathcal{X} such that for every $x \in K$, we can provide an integer $1 \leq N_x \leq \bar{N}$, depending on x , such that for every $0 < \delta < \delta_0$ we have $D_{x,\bar{N}\delta_0} \subset \Omega$ and (27) holds. \square

Corollary 4.2. *Under the assumptions of Lemma 4.1, we have*

$$(29) \quad \inf_{B_{x,b\delta}} u \geq \begin{cases} 2u(x) - (2^{\bar{N}} - 1) \sup_{B_{x,\bar{N}\delta}} u & \text{if } u(x) \geq 0 \\ 2^{\bar{N}} u(x) - (2^{\bar{N}} - 1) \sup_{B_{x,\bar{N}\delta}} u & \text{if } u(x) < 0 \text{ and } \sup_{B_{x,\bar{N}\delta}} u \geq 0 \\ 2^{\bar{N}} u(x) - \sup_{B_{x,\bar{N}\delta}} u & \text{if } \sup_{B_{x,\bar{N}\delta}} u < 0 \end{cases}.$$

The previous corollary immediately leads us to another consequence.

Corollary 4.3. *Every \mathcal{X} -convex function that is locally bounded from above on an open set is also locally bounded from below.*

The proof of Theorem 1.2 follows the scheme of Lemma 3.1 in [20]. In the sequel, we will use the distance function $\text{dist}_d(A, x) = \inf_{a \in A} d(a, x)$ for every $A \subset \mathbb{R}^n$.

Proof of Theorem 1.2. First of all, from Corollary 4.3 it follows that u is locally bounded. Let us choose $0 < D < \text{dist}_d(K, \Omega^c)$ and consider the compact set

$$K_D = \{z \in \mathbb{R}^n : \text{dist}_d(K, z) \leq D\},$$

that is clearly contained in Ω . Choose any $\alpha > 0$ such that $D + \alpha < \text{dist}_d(K, \Omega^c)$. Therefore for every $x \in K_D$ and $X \in \mathcal{X}$, we have

$$\text{dist}_d(K_D, \Phi^X(x, t)) \leq d(\Phi^X(x, t), x) \leq |t| \leq \alpha$$

hence $\Phi^X(x, t) \in K_{D+\alpha} = \{z \in \mathbb{R}^n : \text{dist}_d(K, z) \leq D + \alpha\} \subset \Omega$ for all $|t| \leq \alpha$. Hence Φ^X is defined on $K_D \times [-\alpha, \alpha]$ and it is contained in the larger compact set $K_{D+\alpha} \subset \Omega$. Let us fix $x, y \in K$ such that $\rho(x, y) < D$. Let $\varepsilon > 0$ be arbitrary chosen such that $\rho(x, y) + \varepsilon < D$. Thus, by definition of ρ , there exists $\rho(x, y) < \bar{t} < \rho(x, y) + \varepsilon$ and $\gamma \in \Gamma_{x,y}^c(\bar{t})$ such that $t_0 = 0 < t_1 < \dots < t_\nu = \bar{t}$ and

$$(30) \quad \gamma(t) = \Phi^{X_{j_k}}(\gamma(t_{k-1}), t - t_{k-1})$$

for all $t \in [t_{k-1}, t_k]$ and $k = 1, \dots, \nu$, where $1 \leq j_1, \dots, j_\nu \leq m$. We have that

$$d(\gamma(t), x) \leq \rho(\gamma(t), x) \leq t \leq \bar{t} < D,$$

therefore the whole curve γ is contained in K_D and any restriction $\gamma|_{[t_{k-1}, t_k]}$ can be smoothly extended on $[t_{k-1} - \alpha, t_k + \alpha]$ preserving the same form (30). Since u is locally bounded, we set

$$M = \sup_{w \in K_{D+\alpha}} |u(w)| < +\infty.$$

As a result, the \mathcal{X} -convexity of u implies that the difference quotient

$$\frac{|u(\gamma(t_k)) - u(\gamma(t_{k-1}))|}{|t_k - t_{k-1}|}$$

is not greater than the maximum between $|u(\Phi^{X_{j_k}}(\gamma(t_{k-1}), t_k + \alpha - t_{k-1})) - u(\gamma(t_k))| \alpha^{-1}$ and $|u(\Phi^{X_{j_k}}(\gamma(t_{k-1}), -\alpha)) - u(\gamma(t_{k-1}))| \alpha^{-1}$. This yields proves that

$$\frac{|u(\gamma(t_k)) - u(\gamma(t_{k-1}))|}{|t_k - t_{k-1}|} \leq \frac{2M}{\alpha}.$$

It follows that

$$|u(y) - u(x)| \leq \sum_{k=1}^{\nu} |u(\gamma(t_k)) - u(\gamma(t_{k-1}))| \leq \frac{2M}{T} \sum_{k=1}^{\nu} (t_k - t_{k-1}) < \frac{2M}{\alpha} (\rho(x, y) + \varepsilon),$$

with an arbitrary choice of $\varepsilon > 0$. In the case $\rho(x, y) \geq D$, we immediately have $|u(x) - u(y)| \leq 2M\rho(x, y)/D$, that leads to the inequality

$$|u(x) - u(y)| \leq \frac{2\rho(x, y)}{\min\{D, \alpha\}} \sup_{K_{D+\alpha}} |u|$$

for every $x, y \in K$, where $D, \alpha > 0$ satisfy $D + \alpha < \text{dist}_d(K, \Omega^c)$. Thus, we choose $r = 2D = 2\alpha < \text{dist}_d(K, \Omega^c)$. By Proposition 3.5, it follows that there exists a constant $C > 0$, depending on K , such that $4\rho(x, y) \leq C d(x, y)$ for all $x, y \in K$, hence concluding the proof. \square

Remark 4.4. The Lipschitz estimate (6) restated with respect to the distance ρ has only explicit constants. Precisely, under the assumptions of Theorem 1.2 we have

$$|u(x) - u(y)| \leq \frac{2\rho(x, y)}{\min\{\alpha_1, \alpha_2\}} \sup_{K_{\alpha_1+\alpha_2}} |u|.$$

5. \mathcal{L} -WEAK SUBSOLUTIONS AND UPPER ESTIMATES

The point of this section is to show that locally bounded above \mathcal{X} -convex functions are \mathcal{L} -weak subsolutions of (15), where $\mathcal{X} = \{X_1, \dots, X_m\}$ is a family of Hörmander vector fields. This will enable us to apply the following well known result.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let \mathcal{X} be a family of smooth Hörmander vector fields and let $p > 0$. Thus, there exists $r_0 > 0$, depending on Ω and \mathcal{X} , and there exists $\kappa \geq 1$, depending on p , Ω and \mathcal{X} , such that whenever $u \in W_{\mathcal{X}}^{1,2}(\Omega)$ is a weak \mathcal{L} -subsolution to (15), we have*

$$(31) \quad \operatorname{esssup}_{B_{x,\frac{r}{2}}} u \leq \kappa \left(\int_{B_{x,r}} |u(y)|^p dy \right)^{\frac{1}{p}},$$

for every $x \in \Omega$ such that $0 < r \leq \min\{r_0, \operatorname{dist}(\Omega^c, x)\}$.

The proof of this theorem is standard: it follows the celebrated Moser iteration technique for weak solutions to elliptic equations in divergence form [22], that applies to very general frameworks, including Carnot-Carathéodory spaces. There is a plenty of works in this area, so we limit ourselves to mention just a few of them, [7] [18], [15], [3]. Further discussion of this topic can be found for instance in [14].

In the proof of Theorem 1.3, we will use the following basic fact.

Lemma 5.2. *Let X be a vector field on \mathbb{R}^n , let $z \in \mathbb{R}^n$ be such that $X(z) \neq 0$ and let π be a hyperplane of \mathbb{R}^n transversal to $X(z)$ and passing through z . There exists an open neighbourhood A of z in π , $\tau > 0$ and an open neighbourhood U of z in \mathbb{R}^n such that the restriction of the flow Φ^X to $A \times (-\tau, \tau)$ is a diffeomorphism onto U . Moreover, for every fixed system of coordinates $(\xi_1, \dots, \xi_{n-1})$ on π , denoting by ϕ the previous restriction with respect to these coordinates and by J_ϕ its Jacobian, we get*

$$(32) \quad \operatorname{div} X(x) = \frac{\partial_t J_\phi}{J_\phi} \circ \phi^{-1}(x) \quad \text{for all } x \in U.$$

Remark 5.3. From the definition of commutator and the fact that the family \mathcal{X} satisfies the Hörmander condition, it is clear that for each $z \in \mathbb{R}^n$, there exists $X \in \mathcal{X}$ such that $X(z) \neq 0$.

Proof of Theorem 1.3. As observed in Remark 5.3, since \mathcal{X} is a family of Hörmander vector fields, we must have some $j_1 \in \{1, 2, \dots, m\}$ such that $X_{j_1}(x_0) \neq 0$. Thus, for each $i = 1, \dots, m$, we define $Y_i = X_i$ if $X_i(x_0) \neq 0$ and $Y_i = X_i + X_{j_1}$ otherwise, so that all Y_i do not vanish on x_0 . In view of Lemma 5.2, for each $i = 1, \dots, m$ we can find an open bounded neighbourhood U_i of x_0 , that is compactly contained in Ω , an open bounded set $A_i \subset \mathbb{R}^{n-1}$, $\tau_i > 0$ and a diffeomorphism $\phi_i : S_i \rightarrow U_i$, with $S_i = A_i \times (-\tau_i, \tau_i)$, ϕ_i is the restriction of the flow of Y_i and then it satisfies (32). We can find $\delta_0 > 0$ such that B_{x_0, δ_0} is compactly contained in U_i for all $i = 1, \dots, m$. Let us choose any $\varphi \in C_c^\infty(B_{x_0, \delta_0})$ with $\varphi \geq 0$. Our claim follows if we prove that

$$(33) \quad \sum_{i=1}^m \int_{B_{x_0, \delta_0}} Y_i u(x) Y_i^* \varphi(x) dx \geq 0.$$

We will prove a stronger fact, namely, the validity of

$$\int_{B_{x_0, \delta_0}} Y_i u(x) Y_i^* \varphi(x) dx \geq 0 \quad \text{for all } i = 1, \dots, m.$$

By definition of \mathcal{X} -convexity, we have that $u(\phi_i(\omega, \cdot))$ is convex on the interval where it is defined for all $i = 1, \dots, m$. By Theorem 1.2, u is locally Lipschitz continuous with respect to d . Iterating Lemma 2.6, no more than $m-1$ times, and observing that $\mathcal{X}_1 = \{Y_1, \dots, Y_m\}$ is also a family of Hörmander vector fields, its associated distance d_1 is equivalent to d , that is obtained from \mathcal{X} . Theorem 2.9 and Proposition 2.10 imply that $u \in W_{\mathcal{X}, \text{loc}}^{1, \infty}(\Omega)$ and the pointwise derivative

$$\partial_{Y_i} u(x) = \frac{d}{dt} u(\Phi^{Y_i}(x, t))|_{t=0}$$

exists for almost every $x \in \Omega$ and coincides with the distributional derivative $Y_i u$, up to a negligible set. In particular, there exists $L > 0$ such that $|Y_i u| \leq L$ almost everywhere in U_i , where $Y_i u$ is the distributional derivative of u along Y_i . Since ϕ_i sends negligible sets into negligible sets, we have that

$$(34) \quad \frac{\partial}{\partial s} u(\phi_i(\omega, s))|_{s=t} = \partial_{Y_i} u(\phi(\omega, t)) = Y_i u(\phi_i(\omega, t))$$

for almost every $(\omega, t) \in S_i$. There exist $0 < t_i < \tau_i$ such that $\phi(A_i \times (-t_i, t_i)) = U'_i$ still contains B_{x_0, δ_0} , hence for $\varepsilon > 0$ sufficiently small, we can consider

$$(u \circ \phi_i)_\varepsilon(\omega, t) = \int_{-\tau_i}^{\tau_i} (u \circ \phi_i)(\omega, s) \nu_\varepsilon(t-s) ds,$$

for all $t \in (-t_i, t_i)$, where ν_ε are one dimensional mollifiers. Since $(u \circ \phi_i)(\omega, \cdot)$ is convex on $(-\tau_i, \tau_i)$ it is also locally Lipschitz, with distributional derivative. It follows that

$$\frac{\partial}{\partial t} (u \circ \phi_i)_\varepsilon(\omega, t) = (\partial_{Y_i} u \circ \phi_i)_\varepsilon(\omega, t)$$

for all $\omega \in A_i$ and $t \in (-t_i, t_i)$. Due to (34), applying Fubini's theorem it follows that for almost every $\omega \in A_i$ the pointwise derivative $\partial_{Y_i} u(\omega, t)$ equals the distributional derivative $Y_i u(\omega, t)$ for almost every $t \in (-\tau_i, \tau_i)$, that is precisely represented almost everywhere. As a consequence, we have

$$(35) \quad \frac{\partial}{\partial t} (u \circ \phi_i)_\varepsilon(\omega, t) = (\partial_{Y_i} u \circ \phi_i)_\varepsilon(\omega, t) = ((Y_i u) \circ \phi_i)_\varepsilon(\omega, t)$$

for almost every $\omega \in A_i$ and every $t \in (-t_i, t_i)$. Since $(u \circ \phi)_\varepsilon(\omega, \cdot)$ is smooth and convex for all $\omega \in A_i$, we achieve

$$\int_{S'_i} \frac{\partial^2}{\partial t^2} (u \circ \phi_i)_\varepsilon(\omega, t) \varphi(\phi(\omega, t)) J_{\phi_i}(\omega, t) d\omega dt \geq 0$$

where $S'_i = A_i \times (-t_i, t_i)$. Integrating by parts, it follows that the previous nonnegative integral equals the following one

$$- \int_{S'_i} \frac{\partial}{\partial t} (u \circ \phi_i)_\varepsilon(\omega, t) \frac{\partial}{\partial t} \{ \varphi(\phi_i(\omega, t)) J_{\phi_i} \} d\omega dt$$

that can be written as follows

$$- \int_{S'_i} \left(\frac{\partial}{\partial t} (u \circ \phi_i)_\varepsilon(\omega, t) \frac{\partial}{\partial t} \varphi(\phi_i(\omega, t)) J_{\phi_i} + \frac{\partial}{\partial t} (u \circ \phi_i)_\varepsilon(\omega, t) (\varphi \circ \phi_i)(\omega, t) \frac{\partial}{\partial t} J_{\phi_i} \right) d\omega dt$$

Clearly, we have $\frac{\partial}{\partial t} (\varphi \circ \phi_i)(\omega, t) = (Y_i \varphi)(\phi_i(\omega, t))$, hence by Lemma 5.2, we obtain

$$- \int_{S'_i} \frac{\partial}{\partial t} (u \circ \phi_i)_\varepsilon(\omega, t) \left((Y_i \varphi)(\phi_i(\omega, t)) + (\operatorname{div} Y_i \circ \phi_i)(\omega, t) (\varphi \circ \phi_i)(\omega, t) \right) J_{\phi_i} d\omega dt \geq 0.$$

We can then pass to the limit as $\varepsilon \rightarrow 0^+$, taking into account that $Y_i u \in L^\infty(U_i)$ and that both (34) and (35) hold, getting

$$- \int_{\phi_i^{-1}(U'_i)} (Y_i u) \circ \phi_i \{ (Y_i \varphi \circ \phi_i + (\operatorname{div} Y_i \circ \phi_i) \varphi \circ \phi_i) \} J_{\phi_i} d\omega dt \geq 0.$$

By a change of variables towards the former coordinates, we obtain

$$- \int_{U'_i} Y_i u(x) \{ (Y_i \varphi)(x) + \operatorname{div} Y_i(x) \varphi(x) \} dx = \int_{B_{x_0, \delta_0}} Y_i u(x) Y_i^* \varphi(x) dx \geq 0,$$

that establishes our claim. \square

As a consequence of both Theorem 5.1 and Theorem 1.3, we get the following consequence.

Corollary 5.4. *Let $\Omega \subset \mathbb{R}^n$ be open and let $p > 0$. If $x \in \Omega$, then there exist $\sigma_x, \delta_x > 0$ and $\kappa_x \geq 1$, depending on x , Ω , p and \mathcal{X} , such that $B_{x, \delta_x} \subset \Omega$, $\sigma_x \leq \delta_x/2$ and whenever $u : \Omega \rightarrow \mathbb{R}$ is \mathcal{X} -convex and locally bounded from above, for all $y \in B_{x, \delta_x/2}$ and $0 < r \leq \sigma_x$, we have*

$$(36) \quad \sup_{B_{y, \frac{r}{2}}} u \leq \kappa_x \left(\int_{B_{y, r}} |u(z)|^p dz \right)^{\frac{1}{p}}.$$

Proof. Let $x \in \Omega$ and consider the corresponding $\delta_x > 0$ given by Theorem 1.3, such that $B_{x, \delta_x} \subset \Omega$ and u is a weak subsolution of (7) where the vector fields Y_j depend on x . In view of Theorem 5.1 applied to the open bounded set B_{x, δ_x} , we get some constants $\kappa_x \geq 1$ and $r_x > 0$, depending on B_{x, δ_x} , p , and the vector fields Y_j , such that there holds

$$(37) \quad \operatorname{ess\,sup}_{B_{y, \frac{r}{2}}} u \leq \kappa_x \left(\int_{B_{y, r}} |u(z)|^p dz \right)^{\frac{1}{p}},$$

for all $0 < r \leq \min\{r_x, \text{dist}(B_{x,\delta_x}^c, y)\}$. Since for all $y \in B_{x,\delta_x/2}$, we have

$$\text{dist}(B_{x,\delta_x}^c, y) \geq \delta_x/2,$$

setting $\sigma_x = \min\{r_x, \frac{\delta_x}{2}\}$, then (37) holds for all $0 < r \leq \sigma_x$ and all $y \in B_{x,\delta_x/2}$. \square

Remark 5.5. Notice that we do not need to use the essential supremum in (36), since \mathcal{X} -convex functions that are locally bounded from above are locally Lipschitz continuous, due to Theorem 1.2.

As a consequence of Corollary 5.4, we can easily establish the following result.

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be open, let $p > 0$ and let $K \subset \Omega$ be compact. Then there exists $\sigma > 0$ and $\kappa \geq 1$, depending on K , Ω , \mathcal{X} and p , such that for every \mathcal{X} -convex function $u : \Omega \rightarrow \mathbb{R}$ that is locally bounded from above and for every $x \in K$, we have $B_{x,\sigma} \subset \Omega$ and there holds*

$$(38) \quad \sup_{B_{x,\frac{r}{2}}} u \leq \kappa \left(\int_{B_{x,r}} |u(z)|^p dy \right)^{\frac{1}{p}} \quad \text{for all } 0 < r \leq \sigma.$$

6. REGULARITY ESTIMATES FOR \mathcal{X} -CONVEX FUNCTIONS

In this section we combine the upper and lower estimates for \mathcal{X} -convex functions, that give the proof of Theorem 1.1.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be open, let $K \subset \Omega$ be compact and let $u : \Omega \rightarrow \mathbb{R}$ be a \mathcal{X} -convex function that is locally bounded from above. Then there exists $C_0 > 0$, $b_0 > 0$ and $N_0 > 1$, depending on K , such that for every $x \in K$ there holds*

$$\sup_{B_{x,r}} |u| \leq C_0 \int_{B_{x,N_0 r}} |u(z)| dz$$

whenever $0 < r < b_0$ and $K_0 = \{z \in \mathbb{R}^n : \text{dist}(K, z) \leq N_0 b_0\} \subset \Omega$.

Proof. By Lemma 4.1, we have $\delta_0 > 0$, $0 < b < 1$ and a positive integer \bar{N} such that for every $y \in K$, we have $D_{y,\bar{N}\delta_0} \subset \Omega$ and there exists with $1 \leq N_y \leq \bar{N}$ such that

$$(39) \quad 2^{N_y} u(y) - (2^{N_y} - 1) \sup_{B_{y,\bar{N}\delta}} u \leq \inf_{B_{y,b\delta}} u$$

for all $0 < \delta < \delta_0$. Let us consider $x \in K$ and any $0 < \delta' < b\delta_0/4$, observing that there exists $x' \in B_{x,\delta'}$ such that

$$u(x') \geq - \int_{B_{x,\delta'}} |u(z)| dz.$$

We clearly have $\inf_{B_{x,\delta'}} u \geq \inf_{B_{x',2\delta'}} u$, hence for some $1 \leq N_{x'} \leq \bar{N}$, we can apply the estimate (39) at x' , getting

$$\inf_{B_{x,\delta'}} u \geq 2^{N_{x'}} u(x') - (2^{N_{x'}} - 1) \sup_{B_{x',\bar{N}\frac{2\delta'}{b}}} u.$$

From the previous inequalities, it follows that

$$\inf_{B_{x,\delta'}} u \geq -2^{\bar{N}} \int_{B_{x,\delta'}} |u(z)| dz - (2^{N_{x'}} - 1) \sup_{B_{x,\bar{N}} \frac{4\delta'}{b}} u.$$

Theorem 5.6 provides $\sigma > 0$ and $\kappa \geq 1$ such that, up to choose $\delta_0 > 0$ possibly smaller, such that $\bar{N}\delta_0 < \sigma/2$, hence $\bar{N}\frac{8\delta'}{b} < \sigma$ and it follows that

$$\inf_{B_{x,\delta'}} u \geq -2^{\bar{N}} \int_{B_{x,\delta'}} |u(z)| dz - (2^{\bar{N}} - 1) \kappa \int_{B_{x,\bar{N}} \frac{8\delta'}{b}} |u(z)| dz.$$

As a consequence of Corollary 3.13, we have $Q_0 > 0$ and $r_0 > 0$ such that

$$|B_{x,\bar{N}\frac{8\delta'}{b}}| \leq 2^{Q_0} \left(\bar{N} \frac{8}{b} \right)^{Q_0} |B_{x,\delta'}|,$$

up to making δ_0 further smaller, namely, satisfying $2\bar{N}\delta_0 < r_0$. It follows that

$$\inf_{B_{x,\delta'}} u \geq -2^{\bar{N}} \left[\kappa + (16)^{Q_0} \left(\frac{\bar{N}}{b} \right)^{Q_0} \right] \int_{B_{x,\bar{N}} \frac{8\delta'}{b}} |u(z)| dz$$

and also

$$\sup_{B_{x,\delta'}} u \leq \kappa 2^{Q_0} \left(\bar{N} \frac{4}{b} \right)^{Q_0} \int_{B_{x,\bar{N}} \frac{8\delta'}{b}} |u(z)| dz,$$

that yield a constant $C_0 > 0$ depending on K , such that

$$\sup_{B_{x,r}} |u| \leq C_0 \int_{B_{x,N_0 r}} |u(z)| dz$$

for every $0 < r < b_0$ and every $x \in K$, with $b_0 = b\delta_0/4$ and $N_0 = \bar{N}\frac{8}{b} > 1$. By the previous requirements on δ_0 , being $N_0 b_0 = 2\delta_0 \bar{N}$, we also have

$$K_0 = \{z \in \mathbb{R}^n : \text{dist}(K, z) \leq N_0 b_0\} \subset \Omega,$$

reaching the conclusion of the proof. \square

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be open, let $K \subset \Omega$ be compact and let $\lambda > 1$. Then there exist $\bar{C} > 0$ and $\bar{Q} > 0$, depending on K and there there exists $\bar{r} > 0$, depending on both K and λ , such that for every $x \in K$ and every $0 < r < \bar{r}$, each \mathcal{X} -convex function $u : \Omega \rightarrow \mathbb{R}$, that is locally bounded from above satisfies the following estimate*

$$(40) \quad \sup_{B_{x,r}} |u| \leq \bar{C} \left(\frac{\lambda + 1}{\lambda - 1} \right)^{\bar{Q}} \int_{B_{x,\lambda r}} |u(z)| dz.$$

Proof. We fix any $\beta > 0$ such that $K_1 = \{z \in \mathbb{R}^n : \text{dist}(K, z) \leq \beta\} \subset \Omega$ and apply Theorem 6.1 to K_1 , getting the corresponding positive constants C_1, b_1 and $N_1 > 1$. We have in particular

$$\{z \in \mathbb{R}^n : \text{dist}(K_1, z) \leq N_1 b_1\} \subset \Omega.$$

Taking $0 < r < \beta/\lambda$, we have $B_{x,\lambda r} \subset K_1$ for all $x \in K$ and fixing $a = (\lambda - 1)/N_1$, it follows that for $0 < r < r_1$ and $r_1 = \min\{b_1/a, \beta/\lambda\}$, the following inequality

$$\sup_{B_{y,ar}} |u| \leq C_1 \int_{B_{y,N_1 ar}} |u(z)| dz$$

holds for all $y \in K_1$. Now, let us fix $x \in K$. Thus, whenever $0 < r < r_1$ we can cover the compact set $D_{x,r}$ with a finite number of balls $B_{x_j,ar}$ centered at points of $D_{x,r}$, hence there exists $x_{j_0} \in D_{x,r}$ such that

$$\sup_{B_{x,r}} |u| \leq \sup_{B_{x_{j_0},ar}} |u|.$$

Since $x_{j_0} \in K_1$ and $ar < b_1$, Theorem 6.1 implies that

$$\sup_{B_{x_{j_0},ar}} |u| \leq C_1 \int_{B_{x_{j_0},N_1 ar}} |u(z)| dz = C_1 \int_{B_{x_{j_0},(\lambda-1)r}} |u(z)| dz.$$

As a result, we have proved that

$$\sup_{B_{x,r}} |u| \leq C_1 \frac{|B_{x,\lambda r}|}{|B_{x_{j_0},(\lambda-1)r}|} \int_{B_{x,\lambda r}} |u(z)| dz \leq C_1 \frac{|B_{x_{j_0},(\lambda+1)r}|}{|B_{x_{j_0},(\lambda-1)r}|} \int_{B_{x,\lambda r}} |u(z)| dz$$

for all $0 < r < r_1$, where r_1 also depends on λ . Finally, we apply Corollary 3.13 to K_0 , getting $r_2 > 0$ and $\bar{Q} > 0$ such that for all $0 < r < \min\{r_1, r_2/\lambda + 1\}$ our claim (40) holds with $\bar{C} = C_1 2^{\bar{Q}}$. \square

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